Radiative corrections to chiral separation effect in QED

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We calculate the leading radiative corrections to the axial current in the chiral separation effect in dense QED in a magnetic field. Contrary to the conventional wisdom, suggesting that the axial current should be exactly fixed by the chiral anomaly relation and is described by the topological contribution on the lowest Landau level in the free theory, we find in fact that the axial current receives nontrivial radiative corrections. The direct calculations performed to the linear order in the external magnetic field show that the nontrivial radiative corrections to the axial current are provided by the Fermi surface singularity in the fermion propagator at nonzero fermion density.

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I. INTRODUCTION

Recently there was a significant interest in the dynamics of relativistic matter in a magnetic field. Assuming that QCD topological fluctuations produce local $\mathcal{P}$ and $\mathcal{CP}$-odd states [1] leading to a chiral chemical potential μ_5, it was suggested that there exists a non-dissipative electric current $\mathbf{j} = e^2\mathbf{B}\mu_5/(2\pi^2)$ in relativistic matter in a magnetic field $\mathbf{B}$ [2–4]. This phenomenon is known in the literature as the chiral magnetic effect (CME). (For a recent review see Ref. [5].) Moreover, the charge-dependent correlations and flow, observed in heavy-ions collisions at RHIC [6–9] and LHC [10], appear to be in a qualitative agreement with the predictions of the CME [11, 12].

Unlike the chiral chemical potential, which is a rather exotic quantity and not so well defined theoretically, the chemical potential μ (associated, for example, with conserved electric or baryon charges) is common in many physical systems. It was shown in Refs. [13–15] that a non-dissipative axial current $\mathbf{j}_5 = e\mathbf{B}\mu/(2\pi^2)$ exists in the equilibrium state of noninteracting massless fermion matter in a magnetic field. This effect is known as the chiral separation effect (CSE) in the literature. (For a brief review, see Sec. 2 in Ref. [5].) In fact, as suggested in Refs. [16, 17], the CSE may lead to a chiral charge separation (i.e., effectively inducing a nonzero chiral chemical potential μ_5) and, thus, trigger the CME even in the absence of topological fluctuations in the initial state.

The approach in Refs. [14, 15] was based on the use of the operator form of the chiral anomaly relation [18]. It is well known that the corresponding relation, calculated at one-loop order, is exact and, as such, it cannot get any higher-order radiative corrections [19]. Therefore, it was argued in [14, 15] that like the chiral anomaly, the one-loop result for the axial current density, $\mathbf{j}_5 = e\mathbf{B}\mu/(2\pi^2)$, should be exact as well.

Since the fermion propagator in a magnetic field depends nonlinearly on the magnetic field, the linear dependence of the axial current on $\mathbf{B}$ calls for a physical explanation. Using an expansion over the Landau levels, it was shown in Ref. [14] that the axial current $\mathbf{j}_5 = e\mathbf{B}\mu/(2\pi^2)$ is topological in nature (see also Ref. [20] for a nice exposition and some details) and is defined by the fermion number density on the lowest Landau level (LLL). Moreover, it was shown [14] that a similar result holds even for massive fermions at finite temperature T, where the axial current equals $\mathbf{j}_5 = e\mathbf{B}n_L(m,T)/(2\pi)$ and $n_L(m,T)$ is the effective one-dimensional (along the direction of magnetic field) fermion number density on the LLL. At zero temperature the axial current is given by $\mathbf{j}_5 = e\mathbf{B}\sqrt{\mu^2 - m^2}/(2\pi^2)$. Of course, in the chiral limit $m \to 0$ this reduces to the same expression for the axial current as derived from the chiral anomaly. Note, however, that the connection between the induced axial current and the anomaly relation is not obvious beyond the chiral limit.

The chiral anomaly is exact as an operator relation, but it contains the divergence of the axial current rather than the current itself. Consequently, to get the axial current from the chiral anomaly one should "integrate" the anomaly and calculate the ground state expectation value of the corresponding operator. Then, the question concerning an "integration constant" in the induced axial current and its dependence on interactions naturally arises. Until now, no conclusive answer to this question was given (e.g., see the discussion in Ref. [5]).

The first studies of the interactions effects were done in Refs. [16, 21–23] in the framework of the dense Nambu–Jona-Lasinio (NJL) model in a magnetic field. Using the Schwinger–Dyson equation for the fermion propagator, it was found [16, 21, 22] that the four-fermion interactions generates a chiral shift parameter Δ. In the chiral limit, this parameter determines a relative shift of the momenta in the dispersion relations for opposite chirality fermions

$k^3 \to k^3 \pm \Delta$, where the momentum k^3 is directed along the magnetic field. The presence of the chiral shift parameter leads to an additional dynamical contribution in the axial current. Unlike the topological contribution in the axial current at the LLL, the dynamical one affects the fermions in all Landau levels, including those around the whole Fermi surface. Further, it was explicitly checked in Ref. [22] that although the axial current gets corrections due to the NJL interactions, the chiral anomaly does not.

Since the NJL model is nonrenormalizable and the chiral anomaly is intimately connected with ultraviolet divergencies, in order to reach a solid conclusion about the presence or absence of higher-order radiative corrections to the axial current, one should consider them in a renormalizable model. In the present paper, assuming that the magnetic field $\mathbf{B}$ is weak and using the expansion in powers of $\mathbf{B}$ up to linear order, the leading radiative corrections to the axial current in QED are calculated. We find that they do not vanish and attribute this result to the singularities in the fermion propagator at the Fermi surface. On the technical side, the $i\epsilon\,\mathrm{sign}(k_0)$ prescription in the fermion propagator, which is the only thing that distinguishes a chemical potential from the time component A_0 of the photon field, plays a crucial role in deriving this result.

This paper is organized as follows. In Sec. II we introduce the model and set up the notation. Also, we discuss some properties of the fermion propagator and the one-loop self-energy in the presence of an external magnetic field and a nonzero density. The calculation of the leading radiative corrections to the axial current is presented in Sec. III. We start from the formal definition of the current in terms of the fermion propagator, use its systematic expansion in powers of the magnetic field, and finally perform the explicit calculations. Our discussion of the results and conclusions are given in Sec. IV. A new form of the Schwinger parametrization for the fermion propagator in the case of a nonzero magnetic field and a nonzero chemical potential, utilized in the main part of the paper, is presented in Appendix A. The details of the calculations of the radiative corrections to the axial current are given in Appendix B.

II. FERMION SELF-ENERGY IN A MAGNETIC FIELD

The Lagrangian density of QED in a magnetic field is given by

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}\left(i\gamma^\nu \mathcal{D}_\nu + \mu\gamma^0 - m\right)\psi + \delta_2\bar{\psi}(i\gamma^\nu\partial_\nu + \mu\gamma^0 + eA^{\mathrm{ext}}_\nu\gamma^\nu)\psi - \delta_m\bar{\psi}\psi, \tag{1}$$

where μ is the fermion chemical potential, the last two terms are counterterms (we use the notation of Ref. [24], but with the opposite sign of the electric charge, $e \to -e$), and the covariant derivative is $\mathcal{D}_\mu = \partial_\mu - ieA_\mu - ieA^{\mathrm{ext}}_\mu$. Without the loss of generality, we assume that the external magnetic field $\mathbf{B}$ points in the $+x_3$ direction and is described by the vector potential in the Landau gauge, $A^{\mathrm{ext}}_\mu = (0, 0, Bx_1, 0)$. Note that the counterterms include the chemical potential μ and the external field A^{ext}_μ.

To leading order in the coupling constant $\alpha = e^2/(4\pi)$, the fermion self-energy in QED is given by

$$\Sigma(x, y) = -4i\pi\alpha\gamma^\mu S(x, y)\gamma^\nu D_{\mu\nu}(x - y), \tag{2}$$

where $S(x, y)$ is the free fermion propagator in magnetic field and $D_{\mu\nu}(x - y)$ is the free photon propagator.

As is well known, the fermion propagator $S(x, y)$ in the presence of an external magnetic field is not translation invariant. It can be written, however, in a form of an overall Schwinger phase (breaking the translation invariance) and a translation invariant function [25], i.e.,

$$S(x, y) = \exp\left[i\Phi(x, y)\right]\bar{S}(x - y), \tag{3}$$

where the Schwinger phase equals $\Phi(x, y) = -eB(x_1 + y_1)(x_2 - y_2)/2$ in the Landau gauge. The Fourier transform of $\bar{S}(x - y)$ is presented in Eq. (A1) in Appendix A. The expression in Eq. (2) implies that the self-energy $\Sigma(x, y)$ has an analogous representation

$$\Sigma(x, y) = \exp\left[i\Phi(x, y)\right]\bar{\Sigma}(x - y), \tag{4}$$

with the same Schwinger phase as in the propagator.

In this study we use the photon propagator in the Feynman gauge. In momentum space, it reads

$$D_{\mu\nu}(q) = -i\frac{g_{\mu\nu}}{q^2_\Lambda} \equiv -i\left(\frac{g_{\mu\nu}}{q_0^2 - \mathbf{q}^2 - m_\gamma^2 + i\epsilon} - \frac{g_{\mu\nu}}{q_0^2 - \mathbf{q}^2 - \Lambda^2 + i\epsilon}\right). \tag{5}$$

Here we introduced a nonzero photon mass m_γ which serves as an infrared regulator at the intermediate stages of calculations. Of course, none of the physical observables should depend on this parameter (see Sec. IV below). (Note

that since the classical paper of Stueckelberg [26], it is well known that, unlike non-Abelian theories, introducing a photon mass causes no problems in an Abelian gauge theory, such as QED.) We will see in Sec. III that the leading radiative corrections are logarithmically divergent in the ultraviolet region. As in Ref. [19], we find that the Feynman regularization of the photon propagator (5) with ultraviolet regularization parameter Λ presents the most convenient way of regularizing the theory.

The Fourier transform of the translation invariant function $\bar{\Sigma}(x-y)$ is given by the following expression:

$$\bar{\Sigma}(p) = -4i\pi\alpha \int \frac{d^4k}{(2\pi)^4}\gamma^\mu \, \bar{S}(k)\gamma^\nu D_{\mu\nu}(k-p), \tag{6}$$

where $\bar{S}(k)$ is the Fourier transform of the translation invariant part of the fermion propagator and $D_{\mu\nu}(q)$ is the photon propagator (5).

To linear order in B, the translation invariant part of the free fermion propagator in the momentum representation has the following structure:

$$\bar{S}(k) = \bar{S}^{(0)}(k) + \bar{S}^{(1)}(k) + \cdots , \tag{7}$$

where $\bar{S}^{(0)}$ is the free fermion propagator in the absence of magnetic field and $\bar{S}^{(1)}$ is the linear in magnetic field part. Both of them are derived in Appendix A by making use of a generalized Schwinger parametrization when the chemical potential is nonzero. The final expressions for $\bar{S}^{(0)}$ and $\bar{S}^{(1)}$ can be also rendered in the following equivalent form:

$$\bar{S}^{(0)}(k) = i\frac{(k_0+\mu)\gamma^0 - \mathbf{k}\cdot\boldsymbol{\gamma} + m}{(k_0+\mu+i\epsilon\,\mathrm{sign}(k_0))^2 - \mathbf{k}^2 - m^2} \tag{8}$$

and

$$\bar{S}^{(1)}(k) = -\gamma^1\gamma^2 eB\frac{(k_0+\mu)\gamma^0 - k_3\gamma^3 + m}{\left[(k_0+\mu+i\epsilon\,\mathrm{sign}(k_0))^2 - \mathbf{k}^2 - m^2\right]^2}. \tag{9}$$

The self-energy at zero magnetic field

$$\bar{\Sigma}^{(0)}(p) = -4i\pi\alpha \int \frac{d^4k}{(2\pi)^4}\gamma^\mu \bar{S}^{(0)}(k)\gamma^\nu D_{\mu\nu}(p-k) \tag{10}$$

determines the counterterms δ_2 and δ_m in Eq. (1). To calculate the self-energy (10), we will use the generalized Schwinger parametrization of the fermion propagator $\bar{S}^{(0)}(k)$, see Eq. (A11) in Appendix A. Such a representation allows a natural separation of the propagator (as well as the resulting self-energy) into the "vacuum" and "matter" parts. The former is very similar to the usual vacuum self-energy in QED in the one-loop approximation. The only difference will be the appearance of $p_0+\mu$ instead of p_0. The matter part is an additional contribution that comes from the δ-function contribution in Eq. (A11). Unlike the vacuum part, the matter one has no ultraviolet divergences and vanishes when $|\mu|<m$.

The explicit expression for the vacuum part reads

$$\bar{\Sigma}^{(0)}_{\mathrm{vac}}(p) = \frac{\alpha}{2\pi}\int_0^1 dx \left\{2m - x\left[(p_0+\mu)\gamma^0 - \mathbf{p}\cdot\boldsymbol{\gamma}\right]\right\} \ln\frac{x\Lambda^2}{(1-x)m^2 + xm_\gamma^2 - x(1-x)\left[(p_0+\mu)^2 - \mathbf{p}^2\right]}. \tag{11}$$

Note that, while the integral over x can be easily calculated, we keep the result in this more compact form. We see that the self-energy (11) becomes identical with the well-known vacuum self-energy in QED in the Feynman gauge after performing the substitution $p_0+\mu \to p_0$ [24]. Further, using Eq. (11), we find that the counterterms in (1) are defined as follows [24]:

$$\delta_2 = \frac{d\bar{\Sigma}^{(0)}_{\mathrm{vac}}(p)}{d\not{P}}\Big|_{\not{P}=m} = -\frac{\alpha}{2\pi}\left(\frac{1}{2}\ln\frac{\Lambda^2}{m^2} + \ln\frac{m_\gamma^2}{m^2} + \frac{9}{4}\right), \tag{12}$$

$$\delta m = m - m_0 = \bar{\Sigma}^{(0)}_{\mathrm{vac}}(p)\Big|_{\not{P}=m} = \frac{3\alpha}{4\pi}m\left(\ln\frac{\Lambda^2}{m^2} + \frac{1}{2}\right), \tag{13}$$

where $P=(p_0+\mu,\mathbf{p})$. Note that the fermion wave function renormalization constant is defined as follows: $Z_2 = 1+\delta_2$.

For completeness, let us calculate the additional matter part of the self-energy due to the filled fermion states given by

$$\bar{\Sigma}^{(0)}_{\rm mat}(p) = -\frac{i\alpha}{\pi^2}\int_{-\mu}^{0} dk_0 \int d^3\mathbf{k}\, \frac{(k_0+\mu)\gamma^0 - \mathbf{k}\cdot\boldsymbol{\gamma} - 2m}{(k_0-p_0)^2 - (\mathbf{k}-\mathbf{p})^2}\; \delta\left[(k_0+\mu)^2 - \mathbf{k}^2 - m^2\right]. \tag{14}$$

After performing the integration over the energy and spatial angular coordinates, we find

$$\begin{aligned}\bar{\Sigma}^{(0)}_{\rm mat}(p) = -\frac{\alpha}{\pi}\int_0^{\sqrt{\mu^2-m^2}} \frac{kdk}{|\mathbf{p}|}\Bigg\{\frac{1}{2}\left(\gamma^0 - \frac{2m}{\sqrt{k^2+m^2}}\right)\ln\frac{(p_0+\mu-\sqrt{m^2+k^2})^2-(k-|\mathbf{p}|)^2}{(p_0+\mu-\sqrt{m^2+k^2})^2-(k+|\mathbf{p}|)^2} \\ - \frac{k(\mathbf{p}\cdot\boldsymbol{\gamma})}{|\mathbf{p}|\sqrt{m^2+k^2}}\left(1+\frac{k^2+\mathbf{p}^2-(p_0+\mu-\sqrt{m^2+k^2})^2}{4k|\mathbf{p}|}\ln\frac{(p_0+\mu-\sqrt{m^2+k^2})^2-(k-|\mathbf{p}|)^2}{(p_0+\mu-\sqrt{m^2+k^2})^2-(k+|\mathbf{p}|)^2}\right)\Bigg\}.\end{aligned} \tag{15}$$

While the remaining integral over the absolute value of the momentum k can be also performed, the result will take a rather complicated form that will not add any clarity.

The linear in the magnetic field correction to the translation invariant part of the fermion self-energy in a magnetic field reads

$$\bar{\Sigma}^{(1)}(p) = -4i\pi\alpha\int\frac{d^4k}{(2\pi)^4}\gamma^\mu \bar{S}^{(1)}(k)\gamma^\nu D_{\mu\nu}(p-k). \tag{16}$$

This correction, which in particular contains a chiral shift parameter term, will be analyzed in detail elsewhere [27]. However, we use this expression for $\bar{\Sigma}^{(1)}(p)$ in the derivation of the leading corrections in the axial current in Sec. III below.

III. THE LEADING RADIATIVE CORRECTIONS TO THE AXIAL CURRENT

The renormalization group invariant axial current density, which is a quantity of the principal interest in the present paper, is given by

$$\langle j_5^3\rangle = -Z_2 {\rm tr}\left[\gamma^3\gamma^5 G(x,x)\right], \tag{17}$$

where $G(x,y)$ is the full fermion propagator and $Z_2 = 1+\delta_2$ is the wave function renormalization constant of the fermion propagator, cf. Eq. (1).

To the first order in the coupling constant $\alpha = e^2/(4\pi)$, the propagator reads

$$G(x,y) = S(x,y) + i\int d^4u d^4v S(x,u)\Sigma(u,v)S(v,y) + i\int d^4u d^4v S(x,u)\Sigma_{\rm ct}(u,v)S(v,y), \tag{18}$$

where $S(x,y)$ is the free fermion propagator in the magnetic field, $\Sigma(u,v)$ is the one-loop fermion self-energy, and $\Sigma_{\rm ct}(u,v)$ is the counterterms contribution to the self-energy. The structure of the counterterms contribution is determined by the last two terms in the Lagrangian density (1).

In this paper, we make use of the weak magnetic field expansion in the calculation of the axial current density. Such an expansion is straightforward to obtain from the general expression in Eq. (17) and the representation (18) for the fermion propagator. For the fermion propagator to linear in B order, we have

$$S(x,y) = \bar{S}^{(0)}(x-y) + ie\int d^4z\, \bar{S}^{(0)}(x-z)\gamma^\nu \bar{S}^{(0)}(z-y)A^{\rm ext}_\nu(z). \tag{19}$$

Further, by making use of Eq. (19), the weak field expansion of the self-energy follows from the definition in Eq. (2). (Note that the photon propagator is independent of the magnetic field to this order.) Combining all pieces together, we can find the complete expression for the leading radiative corrections to the axial current (17) in the approximation linear in the magnetic field. In this framework, the diagrammatical representation for the leading radiative corrections to the axial current are shown in Fig. 1 (for simplicity, we do not display the contributions due to counterterms).

Instead of using the expansion for the free propagator in Eq. (19), we find it much more convenient to utilize the Schwinger form of the fermion propagator (3), which consists of a simple phase, that breaks the translation invariance, and a translation invariant function. Taking into account that the Schwinger phase $\Phi(x,y)$ is linear in magnetic field,

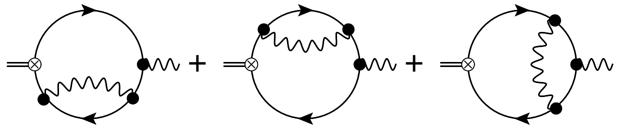

FIG. 1. The leading radiative corrections to the axial current in the linear in magnetic field approximation. Solid and wavy lines correspond to the fermion and photon propagators, respectively. Double solid lines describe the axial current insertions and the external wavy lines attached to the fermion loops indicate the insertions of the external gauge field.

we arrive at the following alternative form of the weak field expansion of the fermion propagator in the linear in B approximation:

$$S(x,y) = \bar{S}^{(0)}(x-y) + i\Phi(x,y)\bar{S}^{(0)}(x-y) + \bar{S}^{(1)}(x-y), \tag{20}$$

where $\bar{S}^{(0)}(x-y)$ and $\bar{S}^{(1)}(x-y)$ are the zeroth and first order terms in powers of B in the translation invariant part of the propagator. [For the explicit forms of their Fourier transforms see Eqs. (8) and (9) above.] Of course, the representations in Eqs. (19) and (20) are equivalent. One can check this explicitly, for example, by making use of the Landau gauge for the external field $A_\nu^{\rm ext}$.

Furthermore, Eq. (2) implies that a similar expansion takes place also for the fermion self-energy

$$\Sigma(u,v) = \bar{\Sigma}^{(0)}(u-v) + i\Phi(u,v)\bar{\Sigma}^{(0)}(u-v) + \bar{\Sigma}^{(1)}(u-v). \tag{21}$$

The Fourier transforms of the self-energies $\bar{\Sigma}^{(0)}(x-y)$ and $\bar{\Sigma}^{(1)}(x-y)$ are given by Eqs. (10) and (16), respectively.

Omitting the noninteresting zeroth order in B contribution in Eq. (18), we arrive at the following linear in B contribution to the propagator:

$$\begin{aligned} G^{(1)}(x,x) = \bar{S}^{(1)}(x,x) &+ i\int d^4u d^4v \left[\bar{S}^{(1)}(x-u)\bar{\Sigma}^{(0)}(u-v)\bar{S}^{(0)}(v-x) + \bar{S}^{(0)}(x-u)\bar{\Sigma}^{(0)}(u-v)\bar{S}^{(1)}(v-x)\right] \\ &+ i\int d^4u d^4v \left[\bar{S}^{(0)}(x-u)\bar{\Sigma}^{(1)}(u-v)\bar{S}^{(0)}(v-x)\right] \\ &- \int d^4u d^4v \left[\Phi(x,u)+\Phi(u,v)+\Phi(v,x)\right]\bar{S}^{(0)}(x-u)\bar{\Sigma}^{(0)}(u-v)\bar{S}^{(0)}(v-x). \end{aligned} \tag{22}$$

Noting that $\Phi(x,u)+\Phi(u,v)+\Phi(v,x) = -\frac{eB}{2}\left[(x_1-u_1)(v_2-x_2)-(v_1-x_1)(x_2-u_2)\right]$ is a translation invariant function, it is convenient to switch to the momentum space on the right hand side of Eq. (22). The result reads

$$\begin{aligned} G^{(1)}(x,x) = &\int \frac{d^4p}{(2\pi)^4}\bar{S}^{(1)}(p) + i\int \frac{d^4p}{(2\pi)^4}\left[\bar{S}^{(1)}(p)\bar{\Sigma}^{(0)}(p)\bar{S}^{(0)}(p) + \bar{S}^{(0)}(p)\bar{\Sigma}^{(0)}(p)\bar{S}^{(1)}(p) + \bar{S}^{(0)}(p)\bar{\Sigma}^{(1)}(p)\bar{S}^{(0)}(p)\right] \\ &- \frac{eB}{2}\int \frac{d^4p}{(2\pi)^4}\left[\frac{\partial\bar{S}^{(0)}(p)}{\partial p_1}\bar{\Sigma}^{(0)}(p)\frac{\partial\bar{S}^{(0)}(p)}{\partial p_2} - \frac{\partial\bar{S}^{(0)}(p)}{\partial p_2}\bar{\Sigma}^{(0)}(p)\frac{\partial\bar{S}^{(0)}(p)}{\partial p_1}\right]. \end{aligned} \tag{23}$$

By substituting this into the definition in Eq. (17), we obtain the following expression for the axial current density:

$$\langle j_5^3\rangle = \langle j_5^3\rangle_0 + \langle j_5^3\rangle_\alpha, \tag{24}$$

where

$$\langle j_5^3\rangle_0 = -\int \frac{d^4p}{(2\pi)^4}\mathrm{tr}\left[\gamma^3\gamma^5\bar{S}^{(1)}(p)\right] \tag{25}$$

is the contribution to the axial current in the free theory and

$$\begin{aligned} \langle j_5^3\rangle_\alpha = &\frac{eB}{2}\int \frac{d^4p}{(2\pi)^4}\mathrm{tr}\left[\gamma^3\gamma^5\frac{\partial\bar{S}^{(0)}(p)}{\partial p_1}\bar{\Sigma}^{(0)}(p)\frac{\partial\bar{S}^{(0)}(p)}{\partial p_2} - \gamma^3\gamma^5\frac{\partial\bar{S}^{(0)}(p)}{\partial p_2}\bar{\Sigma}^{(0)}(p)\frac{\partial\bar{S}^{(0)}(p)}{\partial p_1}\right] \\ &- i\int \frac{d^4p}{(2\pi)^4}\mathrm{tr}\left[\gamma^3\gamma^5\bar{S}^{(1)}(p)\bar{\Sigma}^{(0)}(p)\bar{S}^{(0)}(p) + \gamma^3\gamma^5\bar{S}^{(0)}(p)\bar{\Sigma}^{(0)}(p)\bar{S}^{(1)}(p) + \gamma^3\gamma^5\bar{S}^{(0)}(p)\bar{\Sigma}^{(1)}(p)\bar{S}^{(0)}(p)\right] + \langle j_5^3\rangle_{\rm ct} \end{aligned} \tag{26}$$

defines the leading radiative corrections to the axial current. The counterterms contribution $\langle j_5^3\rangle_{\rm ct}$ in Eq. (26) contains all the contributions with δ_2 and δ_m. Its explicit form will be given in Subsec. III C below.

It is instructive to start from investigating the structure of Eq. (26) in the free theory (i.e., to the zeroth order in α). By making use of the explicit form of $\bar{S}^{(1)}(k)$ in Eq. (9), we straightforwardly derive the following contribution to the axial current density:

$$\langle j_5^3 \rangle_0 = -\frac{eB\,\mathrm{sign}(\mu)}{4\pi^3} \int d^3\mathbf{k}\, \delta(\mu^2 - \mathbf{k}^2 - m^2) = -\frac{eB\,\mathrm{sign}(\mu)}{2\pi^2}\sqrt{\mu^2 - m^2}, \tag{27}$$

which coincides, of course, with the very well known topological contribution [14]. Note that in contrast to the approach using the expansion over the Landau levels, where the contribution to $\langle j_5^3 \rangle_0$ comes only from the filled LLL states, the origin of the same topological contribution in the formalism of weak magnetic fields is quite different. As Eq. (27) implies, it comes from the Fermi surface and, therefore, provides a dual description of the topological contribution in this formalism. (Interestingly, the origin of the topological contribution in the weak field analysis above may have some similarities with the Wigner function formalism [28].)

By substituting the propagators (8) and (9) into Eq. (26), we find the following leading radiative corrections to the axial current:

$$\begin{aligned}
\langle j_5^3 \rangle_\alpha = 32\pi\alpha eB \int \frac{d^4p\, d^4k}{(2\pi)^8} \frac{1}{(P-K)_\Lambda^2} \Bigg[& \frac{(k_0+\mu)[(p_0+\mu)^2 + p_\perp^2 - p_3^2 - m^2] - 2(p_0+\mu)(p_1k_1 + p_2k_2)}{(P^2-m^2)^3(K^2-m^2)} \\
& -2\frac{(p_0+\mu)(p_1k_1+p_2k_2+2k_3p_3+4m^2) - (k_0+\mu)[(p_0+\mu)^2+p_3^2+m^2]}{(P^2-m^2)^3(K^2-m^2)} \\
& -\frac{(k_0+\mu)[(p_0+\mu)^2 - p_\perp^2 + p_3^2 + m^2] - 2(p_0+\mu)p_3k_3}{(P^2-m^2)^2(K^2-m^2)^2} \Bigg] + \langle j_5^3 \rangle_{\rm ct} \\
= 32\pi\alpha eB \int \frac{d^4p\, d^4k}{(2\pi)^8} \frac{1}{(P-K)_\Lambda^2} \Bigg[& \frac{(k_0+\mu)[3(p_0+\mu)^2 + \mathbf{p}^2 + m^2] - 4(p_0+\mu)(\mathbf{p}\cdot\mathbf{k} + 2m^2)}{(P^2-m^2)^3(K^2-m^2)} \\
& -\frac{(k_0+\mu)[3(p_0+\mu)^2 - \mathbf{p}^2 + 3m^2] - 2(p_0+\mu)(\mathbf{p}\cdot\mathbf{k})}{3(P^2-m^2)^2(K^2-m^2)^2} \Bigg] + \langle j_5^3 \rangle_{\rm ct}.
\end{aligned} \tag{28}$$

Here we use the shorthand notation $K^2 = [k_0 + \mu + i\epsilon\,\mathrm{sign}(k_0)]^2 - \mathbf{k}^2$ and $P^2 = [p_0 + \mu + i\epsilon\,\mathrm{sign}(p_0)]^2 - \mathbf{p}^2$. As for the definition of $(P-K)_\Lambda^2$, it follows Eq. (5). Furthermore, the following replacements have been made in the integrand: $p_\perp^2 \to \frac{2}{3}\mathbf{p}^2$, $p_3^2 \to \frac{1}{3}\mathbf{p}^2$, and $p_3k_3 \to \frac{1}{3}(\mathbf{p}\cdot\mathbf{k})$. These replacements are allowed by the rotational symmetry of the other parts of the integrand.

A. Integration by parts

It is convenient to represent Eq. (28) as follows:

$$\begin{aligned}
\langle j_5^3 \rangle_\alpha = 32\pi\alpha eB \int \frac{d^4p d^4k}{(2\pi)^8} \frac{1}{(P-K)_\Lambda^2} \Bigg[& \frac{4(p_0+\mu)[(k_0+\mu)(p_0+\mu) - \mathbf{p}\cdot\mathbf{k} - 2m^2]}{(P^2-m^2)^3(K^2-m^2)} - \frac{(k_0+\mu)}{(P^2-m^2)^2(K^2-m^2)} \\
& -\frac{(k_0+\mu)[3(p_0+\mu)^2 - \mathbf{p}^2 + 3m^2 - 2(\mathbf{p}\cdot\mathbf{k})]}{3(P^2-m^2)^2(K^2-m^2)^2} \Bigg] + \langle j_5^3 \rangle_{\rm ct}.
\end{aligned} \tag{29}$$

Since the denominators of the integrand in this expression contain the factors $(P^2-m^2)^n$ and $(K^2-m^2)^n$, with $n = 2, 3$, which vanish on the Fermi surface, the integrand in (29) is singular there. Therefore, one should carefully treat the singularities in the calculation of the axial current. For this, we find it very convenient to use the following identity valid for all integer $n \geq 1$:

$$\begin{aligned}
\frac{1}{[[k_0+\mu+i\epsilon\,\mathrm{sign}(k_0)]^2 - \mathbf{k}^2 - m^2]^n} = & \frac{1}{[(k_0+\mu)^2 - \mathbf{k}^2 - m^2 + i\epsilon]^n} \\
& + \frac{2\pi i(-1)^{n-1}}{(n-1)!}\theta(|\mu| - |k_0|)\theta(-k_0\mu)\delta^{(n-1)}\left[(k_0+\mu)^2 - \mathbf{k}^2 - m^2\right],
\end{aligned} \tag{30}$$

which can be obtained from Eq. (A5) in Appendix A by differentiating it $n-1$ times with respect to m^2. Since the first term on the right-hand side has the pole prescription as in the theory without the filled fermion states, we call

it the "vacuum" part. The second term in this expression takes care of the filled fermion states, and we call it the "matter" part.

One can also obtain another useful relation by differentiating Eq. (30) with respect to energy k_0,

$$\frac{\partial}{\partial k_0}\left(\frac{1}{[[k_0+\mu+i\epsilon\,\mathrm{sign}(k_0)]^2-m^2-\mathbf{k}^2]^n}\right) = -\frac{2n(k_0+\mu)}{[[k_0+\mu+i\epsilon\,\mathrm{sign}(k_0)]^2-m^2-\mathbf{k}^2]^{n+1}} + \frac{2\pi i(-1)^n\mathrm{sign}(\mu)}{(n-1)!}\delta^{(n-1)}\left[(k_0+\mu)^2-\mathbf{k}^2-m^2\right]\left[\delta(k_0)-\delta(k_0+\mu)\right], \quad (31)$$

where we made use of Eq. (30) the second time, albeit with $n \to n+1$, in order to render the result of derivation on the right hand side in the form of the $(n+1)$th order pole with the conventional $i\epsilon$-prescription at nonzero μ. In addition, we used the following easy to derive result:

$$\frac{\partial}{\partial k_0}\left[\theta(|\mu|-|k_0|)\theta(-k_0\mu)\right] = \mathrm{sign}(\mu)\left[\delta(k_0+\mu)-\delta(k_0)\right]. \quad (32)$$

We note that $\delta(k_0+\mu)$ in the last term on the right hand side of Eq. (31) never contributes. Indeed, this δ-function is nonvanishing only when $k_0+\mu=0$. It multiplies, however, another δ-function, which is nonvanishing only when $(k_0+\mu)^2-\mathbf{k}^2-m^2=0$. Since the two conditions cannot be simultaneously satisfied, the corresponding contribution is trivial. After taking this into account, we finally obtain

$$\frac{\partial}{\partial k_0}\left(\frac{1}{[[k_0+\mu+i\epsilon\,\mathrm{sign}(k_0)]^2-m^2-\mathbf{k}^2]^n}\right) = -\frac{2n(k_0+\mu)}{[[k_0+\mu+i\epsilon\,\mathrm{sign}(k_0)]^2-m^2-\mathbf{k}^2]^{n+1}} + \frac{2\pi i(-1)^n\mathrm{sign}(\mu)}{(n-1)!}\delta^{(n-1)}\left(\mu^2-\mathbf{k}^2-m^2\right)\delta(k_0). \quad (33)$$

Now, by making use of the above identities, we can proceed to the calculation of $\langle j_5^3\rangle_\alpha$ in Eq. (29). We start by simplifying the corresponding expression using integrations by parts. Note that the Λ-regulated representation has nice convergence properties in the ultraviolet and, therefore, all integrations by parts in the analysis that follows will be perfectly justified.

The first term in $\langle j_5^3\rangle_\alpha$ in Eq. (29) is proportional to $p_0+\mu$ and contains $(P^2-m^2)^3$ in the denominator. This suggests to use identity (33) with $n=2$ and $k \to p$, i.e.,

$$\frac{4(p_0+\mu)}{(P^2-m^2)^3} = -\frac{\partial}{\partial p_0}\left(\frac{1}{(P^2-m^2)^2}\right) + 2i\pi\delta'\left[\mu^2-m^2-\mathbf{p}^2\right]\delta(p_0). \quad (34)$$

Using it, we rewrite the first term in the integrand of Eq. (29) as follows:

$$\begin{aligned}
1\mathrm{st} &= f_1 + 32\alpha eB\int\frac{d^4p\,d^4k}{(2\pi)^8}\frac{(k_0+\mu)(p_0+\mu)-\mathbf{p}\cdot\mathbf{k}-2m^2}{(P-K)^2_\Lambda(K^2-m^2)}\frac{\partial}{\partial p_0}\left(\frac{-1}{(P^2-m^2)^2}\right)\\
&= f_1 + 32\alpha eB\int\frac{d^4p\,d^4k}{(2\pi)^8}\frac{1}{(P^2-m^2)^2}\frac{\partial}{\partial p_0}\left(\frac{(k_0+\mu)(p_0+\mu)-\mathbf{p}\cdot\mathbf{k}-2m^2}{(P-K)^2_\Lambda(K^2-m^2)}\right)\\
&= f_1 + 32\alpha eB\int\frac{d^4p\,d^4k}{(2\pi)^8}\frac{1}{(P^2-m^2)^2}\left(\frac{(k_0+\mu)}{(P-K)^2_\Lambda(K^2-m^2)} + \frac{(k_0+\mu)(p_0+\mu)-\mathbf{p}\cdot\mathbf{k}-2m^2}{(K^2-m^2)}\frac{\partial}{\partial p_0}\frac{1}{(P-K)^2_\Lambda}\right),
\end{aligned} \quad (35)$$

where the singular "matter" term, containing the derivative of a δ-function at the Fermi surface, was separated into a new function,

$$f_1 = 64i\pi^2\alpha eB\int\frac{d^4p\,d^4k}{(2\pi)^8}\frac{(k_0+\mu)(p_0+\mu)-\mathbf{p}\cdot\mathbf{k}-2m^2}{(P-K)^2_\Lambda(K^2-m^2)}\delta'\left[\mu^2-m^2-\mathbf{p}^2\right]\delta(p_0). \quad (36)$$

We note that the first term in the parenthesis in Eq. (35) cancels with the second term in the integrand of Eq. (29). Then using

$$\frac{\partial}{\partial p_0}\frac{1}{(P-K)^2_\Lambda} = -\frac{\partial}{\partial k_0}\frac{1}{(P-K)^2_\Lambda} \quad (37)$$

and integrating by parts, we find that the sum of first and second terms in the integrand of Eq. (29) is equal to

$$(1\text{st}+2\text{nd}) = f_1 + 32\alpha eB\int\frac{d^4pd^4k}{(2\pi)^8}\frac{1}{(P-K)^2_\Lambda(P^2-m^2)^2}\left(\frac{p_0+\mu}{(K^2-m^2)}+\left[(k_0+\mu)(p_0+\mu)-\mathbf{p}\cdot\mathbf{k}-2m^2\right]\frac{\partial}{\partial k_0}\frac{1}{(K^2-m^2)}\right)$$
$$= f_1 + f_2 + 32\alpha eB\int\frac{d^4pd^4k}{(2\pi)^8}\frac{1}{(P-K)^2_\Lambda}\left(\frac{(p_0+\mu)}{(P^2-m^2)^2(K^2-m^2)}-2(k_0+\mu)\frac{(k_0+\mu)(p_0+\mu)-\mathbf{p}\cdot\mathbf{k}-2m^2}{(P^2-m^2)^2(K^2-m^2)^2}\right). \tag{38}$$

Note that here we used the identity

$$\frac{\partial}{\partial k_0}\left(\frac{1}{K^2-m^2}\right)=\frac{-2(k_0+\mu)}{(K^2-m^2)^2}-2i\pi\delta\left(\mu^2-m^2-\mathbf{k}^2\right)\delta(k_0), \tag{39}$$

which follows from Eq. (33) with $n=1$, and introduced another function, which contains the leftover contribution with the δ-function,

$$f_2=-64i\pi^2\alpha eB\int\frac{d^4pd^4k}{(2\pi)^8}\frac{(k_0+\mu)(p_0+\mu)-\mathbf{p}\cdot\mathbf{k}-2m^2}{(P-K)^2_\Lambda(P^2-m^2)^2}\delta\left(\mu^2-m^2-\mathbf{k}^2\right)\delta(k_0). \tag{40}$$

It is convenient to make the change of variables $p\to k$ and $k\to p$ in the first term in Eq. (38). Then, the two terms in the integrand can be combined, resulting in

$$(1\text{st}+2\text{nd}) = f_1+f_2+32\alpha eB\int\frac{d^4pd^4k}{(2\pi)^8}\frac{(k_0+\mu)\left[-(p_0+\mu)^2-\mathbf{p}^2+2\mathbf{p}\cdot\mathbf{k}+3m^2\right]}{(P-K)^2_\Lambda(P^2-m^2)^2(K^2-m^2)^2}. \tag{41}$$

Finally, by combining the result in Eq. (41) with the last term in the integrand of Eq. (29), we obtain

$$\langle j_5^3\rangle_\alpha = f_1+f_2-\frac{64}{3}\pi\alpha eB\int\frac{d^4pd^4k}{(2\pi)^8}\frac{(k_0+\mu)}{(P-K)^2_\Lambda}\frac{3(P^2-m^2)+4\mathbf{p}\cdot(\mathbf{p}-\mathbf{k})}{(P^2-m^2)^2(K^2-m^2)^2}+\langle j_5^3\rangle_{\text{ct}}. \tag{42}$$

Using the identity in Eq. (39) once again, we rewrite the last expression as follows:

$$\begin{aligned}\langle j_5^3\rangle_\alpha &= f_1+f_2+f_3+\langle j_5^3\rangle_{\text{ct}}\\ &\quad+\frac{64}{3}\pi\alpha eB\int\frac{d^4pd^4k}{(2\pi)^8}\frac{(k_0-p_0)}{(P-K)^4_\Lambda}\left(\frac{3}{(P^2-m^2)(K^2-m^2)}+\frac{4\mathbf{p}\cdot(\mathbf{p}-\mathbf{k})}{(P^2-m^2)^2(K^2-m^2)}\right),\end{aligned} \tag{43}$$

where

$$f_3=\frac{64i\pi^2\alpha eB}{3}\int\frac{d^4pd^4k}{(2\pi)^8}\frac{3(P^2-m^2)+4\mathbf{p}\cdot(\mathbf{p}-\mathbf{k})}{(P-K)^2_\Lambda(P^2-m^2)^2}\delta\left(\mu^2-m^2-\mathbf{k}^2\right)\delta(k_0). \tag{44}$$

Since the first term of the integrand in the second line of Eq. (43) is odd under the exchange $p\leftrightarrow k$, its contribution vanishes, and we obtain

$$\langle j_5^3\rangle_\alpha = f_1+f_2+f_3+\langle j_5^3\rangle_{\text{ct}}+\frac{64}{3}\pi\alpha eB\int\frac{d^4pd^4k}{(2\pi)^8}\frac{(k_0-p_0)}{(P-K)^4_\Lambda}\frac{4\mathbf{p}\cdot(\mathbf{p}-\mathbf{k})}{(P^2-m^2)^2(K^2-m^2)}. \tag{45}$$

Finally, by making use of the identity

$$\frac{\mathbf{p}\cdot(\mathbf{p}-\mathbf{k})}{(P-K)^4_\Lambda}=\frac{1}{2}\mathbf{p}\cdot\boldsymbol{\nabla}_{\mathbf{k}}\frac{-1}{(P-K)^2_\Lambda} \tag{46}$$

and integrating by parts, we derive

$$\begin{aligned}\langle j_5^3\rangle_\alpha &= f_1+f_2+f_3+\langle j_5^3\rangle_{\text{ct}}+\frac{64}{3}\pi\alpha eB\int\frac{d^4pd^4k}{(2\pi)^8}\frac{2(k_0-p_0)}{(P^2-m^2)^2(K^2-m^2)}\mathbf{p}\cdot\boldsymbol{\nabla}_{\mathbf{k}}\frac{-1}{(P-K)^2_\Lambda}\\ &= f_1+f_2+f_3+\langle j_5^3\rangle_{\text{ct}}+\frac{64}{3}\pi\alpha eB\int\frac{d^4pd^4k}{(2\pi)^8}\frac{2(k_0-p_0)}{(P-K)^2_\Lambda(P^2-m^2)^2}\mathbf{p}\cdot\boldsymbol{\nabla}_{\mathbf{k}}\frac{1}{(K^2-m^2)}\\ &= f_1+f_2+f_3+\langle j_5^3\rangle_{\text{ct}}+\frac{64}{3}\pi\alpha eB\int\frac{d^4pd^4k}{(2\pi)^8}\frac{4(k_0-p_0)\mathbf{p}\cdot\mathbf{k}}{(P-K)^2_\Lambda(P^2-m^2)^2(K^2-m^2)^2}\\ &= f_1+f_2+f_3+\langle j_5^3\rangle_{\text{ct}},\end{aligned} \tag{47}$$

where the last integral term in the penultimate line of Eq. (47) vanishes because it is odd under the exchange $p \leftrightarrow k$. Collecting together all contributions, i.e., f_1 in Eq. (36), f_2 in Eq. (40) and f_3 in Eq. (44), we have the following leading radiative corrections to the axial current:

$$\langle j_5^3 \rangle_\alpha = 64 i \pi^2 \alpha e B \int \frac{d^4 p d^4 k}{(2\pi)^8} \left[\frac{(k_0+\mu)(p_0+\mu) - \mathbf{p}\cdot\mathbf{k} - 2m^2}{(P-K)^2_\Lambda (K^2 - m^2)} \delta' \left[\mu^2 - m^2 - \mathbf{p}^2\right] \delta(p_0) \right.$$
$$\left. + \frac{3(p_0+\mu)^2 - 3(k_0+\mu)(p_0+\mu) + \mathbf{p}^2 - \mathbf{p}\cdot\mathbf{k} + 3m^2}{3(P-K)^2_\Lambda (P^2-m^2)^2} \delta\left(\mu^2 - m^2 - \mathbf{k}^2\right) \delta(k_0) \right] + \langle j_5^3 \rangle_{\text{ct}}, \qquad (48)$$

where the first term in the integrand comes from f_1, while the second term comes from the sum $f_2 + f_3$. The result in Eq. (48) is quite remarkable for several reasons. From a technical viewpoint, it reveals that the integration by parts allowed us to reduce the original two-loop expression in Eq. (29) down to a much simpler one-loop form. Indeed, after the integration over one of the momenta in Eq. (48) is performed using the δ-functions in the integrand, the expression will have an explicit one-loop form. Such a simplification will turn out to be extremely valuable, allowing us to obtain an analytic result for the leading radiative corrections to the axial current.

In addition, the result in Eq. (48) reveals important physics details about the origin of the radiative corrections to the axial current. It shows that all nonzero corrections come from the regions of the phase space, where either p or k momentum is restricted to the Fermi surface. This resembles the origin of the topological contribution in Eq. (27). In both cases, the presence of the singular "matter" terms in identities like (34) and (39) was crucial for obtaining a nonzero result. Moreover, by tracing back the derivation of the result in Eq. (48), we see that all nonsingular terms are gone after the integration by parts. This makes us to conclude that the nonzero radiative corrections to the axial current are intimately connected with the precise form of the singularities in the fermion propagator at the Fermi surface, that separates the filled fermion states with energies less than μ and empty states with larger energies.

B. Counterterms contribution

The calculation of the axial current in Eq. (48) is still technically quite involved. However, it is relatively straightforward to show [see also the derivation of Eq. (B7) in Appendix B] that the right-hand side in (48) without the counterterm has a logarithmically divergent contribution when $\Lambda \to \infty$, i.e.,

$$\frac{\alpha e B (2\mu^2 + m^2)}{4\pi^3 \sqrt{\mu^2 - m^2}} \ln \frac{\Lambda}{m}. \qquad (49)$$

To cancel this divergence, we should add the contribution due the counterterms in Lagrangian (1). The Fourier transforms of the translational invariant part of the counterterm contribution to the self-energy reads

$$\bar{\Sigma}^{(0)}_{\text{ct}}(p) = \delta_2 [(p_0+\mu)\gamma^0 - \mathbf{p}\cdot\boldsymbol{\gamma}] - \delta_m, \qquad (50)$$

where δ_2 was defined in Eq. (12), while $\delta_m = Z_2 m_0 - m \simeq m\delta_2 - \delta m$ and δm was defined in Eq. (13).

We find the following leading order contributions to the axial current density due to counterterms:

$$\begin{aligned}
\langle j_5^3 \rangle_{\text{ct}} &= -\delta_2 \langle j_5^3 \rangle_0 - 4ieB \int \frac{d^4 p}{(2\pi)^4} \frac{\delta_2 (p_0+\mu)}{(P^2-m^2)^2} - 8ieB \int \frac{d^4 p}{(2\pi)^4} \frac{(p_0+\mu)\left[\delta_2((p_0+\mu)^2 - \mathbf{p}^2 + m^2) - 2m\delta_m\right]}{(P^2-m^2)^3} \\
&= -8ieB \int \frac{d^4 p}{(2\pi)^4} \frac{(p_0+\mu)\left[\delta_2 (P^2 - m^2) + 2m(m\delta_2 - \delta_m)\right]}{(P^2-m^2)^3} \\
&= -8ieB\delta_2 \int \frac{d^4 p}{(2\pi)^4} \frac{p_0+\mu}{(P^2-m^2)^2} - 8im\,(m\delta_2 - \delta_m)\, eB \frac{\partial}{\partial(m^2)} \int \frac{d^4 p}{(2\pi)^4} \frac{p_0+\mu}{(P^2-m^2)^2} \\
&= -\frac{eB}{\pi^2}\sqrt{\mu^2-m^2}\,\delta_2 + \frac{eBm\,(m\delta_2 - \delta_m)}{2\pi^2 \sqrt{\mu^2-m^2}}.
\end{aligned} \qquad (51)$$

Here we used the same result of integration as in the topological term, see Eq. (27).

By making use of the explicit form of the counterterms (12) and (13), we obtain

$$\langle j_5^3 \rangle_{\text{ct}} = -\frac{\alpha e B}{2\pi^3} \sqrt{\mu^2 - m^2} \left(\frac{1}{2} \ln \frac{\Lambda^2}{m^2} + \ln \frac{m_\gamma^2}{m^2} + \frac{9}{4} \right) - \frac{3\alpha e B m^2}{4\pi^3 \sqrt{\mu^2 - m^2}} \left(\frac{1}{2} \ln \frac{\Lambda^2}{m^2} + \frac{1}{4} \right). \qquad (52)$$

For $m \ll |\mu|$, it reduces to

$$\langle j_5^3 \rangle_{\rm ct} \simeq -\frac{\alpha eB\mu}{2\pi^3}\left(\frac{1}{2}\ln\frac{\Lambda^2}{m^2} + \ln\frac{m_\gamma^2}{m^2} + \frac{9}{4}\right) - \frac{\alpha eBm^2}{2\pi^3\mu}\left(\frac{1}{2}\ln\frac{\Lambda^2}{m_\gamma^2} - \frac{3}{4}\right). \tag{53}$$

C. The final result

The complete expression for the leading radiative corrections to the axial current is given by Eq. (47). It consists of the counterterm contribution, calculated in the previous subsection, and the additional matter contribution $f_1+f_2+f_3$. The latter is calculated in Appendix B. For $m \ll |\mu|$, it reads

$$f_1 + f_2 + f_3 = \frac{\alpha eB\mu}{2\pi^3}\left(\ln\frac{\Lambda}{2\mu} + \frac{11}{12}\right) + \frac{\alpha eBm^2}{2\pi^3\mu}\left(\ln\frac{\Lambda}{2^{3/2}\mu} + \frac{1}{6}\right). \tag{54}$$

Note that this expression has the right ultraviolet logarithmic divergencies (when $\Lambda \to \infty$) that will cancel exactly with those in the counterterm (53). Combining the two results, we finally obtain the following leading radiative corrections to the axial current in the case $m \ll |\mu|$:

$$\langle j_5^3 \rangle_\alpha = -\frac{\alpha eB\mu}{2\pi^3}\left(\ln\frac{2\mu}{m} + \ln\frac{m_\gamma^2}{m^2} + \frac{4}{3}\right) - \frac{\alpha eBm^2}{2\pi^3\mu}\left(\ln\frac{2^{3/2}\mu}{m_\gamma} - \frac{11}{12}\right). \tag{55}$$

As expected, this result is independent of the ultraviolet regulator Λ. It does contain, however, the dependence on the fictitious photon mass m_γ. This is the only infrared regulator left in our result. Its origin can be easily traced back to the infrared singularity of the wave function renormalization Z_2 in the Feynman gauge used. As we discuss in the next section, this singularity is typical for a class of QED observables, obtained by perturbative methods. As we will explain below, in the complete physical expression for the axial current, obtained by going beyond the simplest double expansion in the coupling constant and magnetic field, the regulator m_γ^2 will likely be replaced by a physical scale, e.g., such as $|eB|$ or $\alpha\mu^2$.

IV. DISCUSSIONS AND CONCLUSIONS

Our study of the chiral separation effect in dense QED in the limit of a weak magnetic field suggests a conceptually new way to interpret and calculate the axial current density even in noninteracting theory. In contrast to the original formulation, which suggests that the topological contribution comes exclusively from the LLL filled states [14], we show that the origin of the same contribution in the formalism of weak magnetic fields (27) is quite different: it comes from the whole Fermi surface. Such a dual description of the topological contribution is of interest on its own.

Our result for the axial current density, obtained perturbatively in the coupling constant and in linear order in the external magnetic field, shows that the chiral separation effect in QED has nonvanishing radiative corrections. To leading order, these corrections are shown to be directly connected with the Fermi surface singularities in the fermion propagator at nonzero density. This interpretation is strongly supported by another observation: had we ignored the corresponding singular terms in the fermion propagator, the calculation of the two-loop radiative corrections would give a vanishing result.

The final result for the leading radiative corrections to the axial current density is presented in Eq. (55). This is obtained by a direct calculation of all relevant contributions to linear order in α and to linear order in the external magnetic field (strictly speaking, linear in eB because the field always couples with the charge). The result in Eq. (55) is presented in terms of renormalized (physical) parameters. As expected, it is independent of the ultraviolet regulator Λ, used at intermediate stages of calculations. This is a nontrivial statement since the original two-loop expression for the leading radiative corrections contains ultraviolet divergencies. In fact, the divergencies are unavoidable because the corresponding diagrams contain the insertions of the one-loop self-energy and vertex diagrams, which are known to have logarithmic divergencies. However, at the end of the day, all such divergencies are canceled exactly with the contributions due to the counterterms.

Our analysis shows that the matter contribution, $f_1 + f_2 + f_3$, to the axial current density (calculated in the Feynman gauge) has no additional singularities. While functions f_1 and f_2+f_3 separately do have additional infrared singularities, the physically relevant result for the sum $f_1 + f_2 + f_3$ is finite, see Appendix B for details. As we see from Eq. (55), however, the final result depends on the photon mass m_γ, which was introduced as the conventional infrared regulator. This feature deserves some additional discussion.

It is straightforward to trace the origin of the m_γ dependence in Eq. (55) to the calculation of the well known result for the wave function renormalization constant δ_2, presented in Eq. (12). In fact, this infrared problem is common for dynamics in external fields in QED (for a thorough discussion, see Sec. 14 in book [29]). The most famous example is provided by the calculation of the Lamb shift, when an electron is in a Coulomb field. The point is that even for a light nucleus with $Z\alpha \ll 1$, one cannot consider the Coulomb field as a weak perturbation in deep infrared. The reason is that this field essentially changes the dispersion relation for the electron at low energy and momenta. As a result, its four-momenta are not on the electron mass shell, where the infrared divergence is generated in the renormalization constant Z_2. Because of that, this infrared divergence is fictitious. The correct approach is to consider the Coulomb interaction perturbatively only at high energies, while to treat it nonperturbatively at low energies. The crucial point is matching those two regions that leads to replacing the fictituous parameter m_γ by a physical infrared scale. This is the main subtlety that makes the calculation of the Lamb shift quite involved [29].

In the case of the Lamb shift, the infrared scale is related to the atomic binding energy, or equivalently the inverse Bohr radius. For smaller energies and momenta, the electron wave functions cannot possibly be approximated with plane waves, which is the tacit assumption of the weak field approximation. Almost exactly the same line of arguments applies in the present problem of QED in an external magnetic field. In particular, the fermion momenta perpendicular to the magnetic field cannot be defined with a precision better than $\sqrt{|eB|}$, or equivalently the inverse magnetic length. This implies that the contribution to the axial current, which comes from the low-energy photon exchange between the fermion states near the Fermi surface, should be treated nonperturbatively. Just like in the Lamb shift problem [29], we can anticipate that a proper nonperturbative treatment will result in a term proportional to $\ln(|eB|/m_\gamma^2)$, with a coefficient such as to cancel the m_γ dependence in Eq. (55).

The additional complication in the problem at hand, which is absent in the study of the Lamb shift, is a nonzero density of matter. While doing the expansion in α and keeping only the leading order corrections, we ignored all screening effects, which formally appear to be of higher order. It is understood, however, that such effects can be very important at nonzero density. In particular, they could replace the unphysical infrared regulator m_γ^2 with a physical screening mass, i.e., the Debye mass $\sqrt{\alpha}\mu$ or some combination of powers of α, μ and $\sqrt{|eB|}$. This important issue will be addressed elsewhere.

Another natural question to address is the chiral limit, $m \to 0$. As one can see from Eq. (55), the current $\langle j_5^3 \rangle_\alpha$ is singular in this limit. This point reflects the well known fact that massless QED possesses new types of infrared singularities: Beside the well known divergences connected with soft photons, there are also divergences connected with the emission and absorption of collinear fermion-antifermion pairs [30, 31]. In addition, because of a Gaussian infrared fixed point in massless QED, the renormalized electric charge of massless fermions is completely shielded. One can show that this property is also intimately related to the collinear infrared divergences [32]. The complete screening of the renormalized electric charge makes this theory very different from massive QED. It remains to be examined whether there is a sensible way to describe the interactions with external electromagnetic fields in massless QED [33].

In addition to the quantitative study of the nonperturbative low-energy contributions and the effect of screening, there remain several other interesting problems to investigate in the future. Here we will mention only the following two. (i) It is of special interest to clarify the connection of the nontrivial radiative corrections to the axial current density calculated in this paper with the generation of the chiral shift parameter in dense QED. The preliminary analysis [27] shows that there is indeed such a connection but it is more complicated than that in the NJL model [16, 21, 22]. (ii) The analysis made in the NJL model shows a lot of similarities between the structure of the chiral current in the CSE effect [16, 21, 22] with that of the electromagnetic current in the CME one [23]. It will be interesting, therefore, to study the induced electromagnetic current in the CME effect in QED with a chiral chemical potential μ_5, using the method developed in the present paper.

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Appendix A: Schwinger parametrization for the fermion propagator at $B \neq 0$ and $\mu \neq 0$

The proper-time representation for the fermion propagator in a constant external magnetic field was obtained long time ago by Schwinger [25]. A naive generalization of the corresponding representation to the case of a nonzero chemical potential (or density) does not work however. This is due to the complications in the definition of the causal Feynman propagator in the complex energy plane when $\mu \neq 0$. The correct analytical properties of such a propagator, describing particles above Fermi surface propagating forward in time and holes below Fermi surface propagating backward in time, are implemented by introducing an appropriate $i\epsilon$-prescription. In particular, one replaces $k_0 + \mu$ with $k_0 + \mu + i\epsilon\,\mathrm{sign}(k_0)$, where ϵ is a vanishingly small positive parameter. For example, in the Landau level representation, the Fourier transform of the translation invariant part of the fermion propagator is defined as follows:

$$\bar{S}(k) = ie^{-k_\perp^2\ell^2}\sum_{n=0}^{\infty}\frac{(-1)^n D_n(k)}{[k_0+\mu+i\epsilon\,\mathrm{sign}(k_0)]^2 - m^2 - k_3^2 - 2n|eB|}, \tag{A1}$$

where the residue at each individual Landau level is determined by

$$D_n(k) = 2\left[(k_0+\mu)\gamma^0 + m - k^3\gamma^3\right]\left[P_- L_n\left(2k_\perp^2\ell^2\right) - P_+ L_{n-1}\left(2k_\perp^2\ell^2\right)\right] + 4(\boldsymbol{k}_\perp\cdot\boldsymbol{\gamma}_\perp)L_{n-1}^1\left(2k_\perp^2\ell^2\right), \tag{A2}$$

where $L_n^\alpha(x)$ are associated Laguerre polynominals.

Let us start by reminding the usual Schwinger's proper-time representation at zero fermion density, i.e.,

$$\frac{1}{[k_0+i\epsilon\,\mathrm{sign}(k_0)]^2 - \mathcal{M}_n^2} \equiv \frac{1}{k_0^2 - \mathcal{M}_n^2 + i\epsilon} = -i\int_0^\infty ds e^{is(k_0^2-\mathcal{M}_n^2+i\epsilon)}, \tag{A3}$$

where $\mathcal{M}_n^2 = m^2 + k_3^2 + 2n|eB|$. It is important to emphasize that the convergence of the integral and, thus, the validity of the representation are insured by having the positive parameter ϵ in the exponent. Unfortunately, such a representation fails at finite fermion density. Indeed, by taking into account that

$$\frac{1}{[k_0+\mu+i\epsilon\,\mathrm{sign}(k_0)]^2 - \mathcal{M}_n^2} \equiv \frac{1}{(k_0+\mu)^2 - \mathcal{M}_n^2 + i\epsilon\,\mathrm{sign}(k_0)\mathrm{sign}(k_0+\mu)}, \tag{A4}$$

we see that the sign of the $i\epsilon$-term in the denominator is not fixed any more. The corresponding sign is determined by the product of $\mathrm{sign}(k_0)$ and $\mathrm{sign}(k_0+\mu)$ and can change, depending on the values of k_0 and μ. For example, while it is positive for $|k_0| > |\mu|$, it turns negative when $|k_0| < |\mu|$ and $k_0\mu < 0$. This seemingly innocuous property causes a serious problem for the integral representation utilized in Eq. (A3). The sign changing $i\epsilon$-term in the exponent invalidates the representation at least for a range of quasiparticle energies.

In order to derive a modified proper-time representation for the fermion propagator, we will make use of the following identity:

$$\begin{aligned}
\frac{1}{[k_0+\mu+i\epsilon\,\mathrm{sign}(k_0)]^2 - \mathcal{M}_n^2} &= \frac{\theta(|k_0|-|\mu|)}{(k_0+\mu)^2 - \mathcal{M}_n^2 + i\epsilon} + \theta(|\mu|-|k_0|)\left(\frac{\theta(k_0\mu)}{(k_0+\mu)^2 - \mathcal{M}_n^2 + i\epsilon} + \frac{\theta(-k_0\mu)}{(k_0+\mu)^2 - \mathcal{M}_n^2 - i\epsilon}\right)\\
&= \frac{1}{(k_0+\mu)^2 - \mathcal{M}_n^2 + i\epsilon} - \theta(|\mu|-|k_0|)\theta(-k_0\mu)\left(\frac{1}{(k_0+\mu)^2 - \mathcal{M}_n^2 + i\epsilon} - \frac{1}{(k_0+\mu)^2 - \mathcal{M}_n^2 - i\epsilon}\right)\\
&= \frac{1}{(k_0+\mu)^2 - \mathcal{M}_n^2 + i\epsilon} + 2i\pi\,\theta(|\mu|-|k_0|)\theta(-k_0\mu)\delta\left[(k_0+\mu)^2 - \mathcal{M}_n^2\right].
\end{aligned} \tag{A5}$$

The first term on the right hand side of Eq. (A5) has a vacuum-like $i\epsilon$-prescription and, thus, allows a usual proper-time representation. The second term is singular and represents the additional "matter" piece, which would be lost in the naive proper-time representation. After making use of this identity, we derive the following modified proper-time representation for the propagator:

$$\begin{aligned}
\bar{S}(k) = e^{-k_\perp^2\ell^2}\sum_{n=0}^{\infty}(-1)^n D_n(k)\int_0^\infty ds\, e^{is[(k_0+\mu)^2-m^2-k_3^2-2n|eB|+i\epsilon]}\\
-\theta(|\mu|-|k_0|)\theta(-k_0\mu)e^{-k_\perp^2\ell^2}\sum_{n=0}^{\infty}(-1)^n D_n(k)\Bigg[\int_0^\infty ds\, e^{is[(k_0+\mu)^2-m^2-k_3^2-2n|eB|+i\epsilon]}\\
+\int_0^\infty ds\, e^{-is[(k_0+\mu)^2-m^2-k_3^2-2n|eB|-i\epsilon]}\Bigg].
\end{aligned} \tag{A6}$$

In order to perform the sum over the Landau levels, we use the following result for the infinite sum of the Laguerre polynominals:

$$\sum_{n=0}^{\infty} z^n L_n^{\alpha}(x) = \frac{1}{(1-z)^{1+\alpha}} \exp\left(\frac{xz}{z-1}\right). \tag{A7}$$

Then we obtain

$$\begin{aligned}
\bar{S}(k) = &\int_0^{\infty} ds\, e^{is[(k_0+\mu)^2-m^2-k_3^2+i\epsilon]-ik_\perp^2\ell^2\tan(s|eB|)} \left[(k_0+\mu)\gamma^0 + m - \mathbf{k}\cdot\boldsymbol{\gamma} - (k^1\gamma^2 - k^2\gamma^1)\tan(seB)\right] \\
&\times \left[1+\gamma^1\gamma^2\tan(seB)\right] - \theta(|\mu|-|k_0|)\theta(-k_0\mu) \\
&\times \Bigg\{ \int_0^{\infty} ds e^{is[(k_0+\mu)^2-m^2-k_3^2+i\epsilon]-ik_\perp^2\ell^2\tan(s|eB|)} \left[(k_0+\mu)\gamma^0 + m - \mathbf{k}\cdot\boldsymbol{\gamma} - (k_1\gamma^2 - k_2\gamma^1)\tan(seB)\right] \\
&\times \left[1+\gamma^1\gamma^2\tan(seB)\right] \\
&+ \int_0^{\infty} ds e^{-is[(k_0+\mu)^2-m^2-k_3^2-i\epsilon]+ik_\perp^2\ell^2\tan(s|eB|)} \left[(k_0+\mu)\gamma^0 + m - \mathbf{k}\cdot\boldsymbol{\gamma} + (k_1\gamma^2 - k_2\gamma^1)\tan(seB)\right] \\
&\times \left[1-\gamma^1\gamma^2\tan(seB)\right] \Bigg\}.
\end{aligned} \tag{A8}$$

This is a very convenient alternative representation for the fermion propagator in a constant external magnetic when $\mu \neq 0$. It allows, in particular, a straightforward derivation of the expansion in powers of the magnetic field. To zeroth order in magnetic field, we obtain

$$\bar{S}^{(0)}(k) = \bar{S}^{(0)}_{\mathrm{vac}}(k) + \bar{S}^{(0)}_{\mathrm{mat}}(k), \tag{A9}$$

where

$$\bar{S}^{(0)}_{\mathrm{vac}}(k) = \int_0^{\infty} ds\, e^{is[(k_0+\mu)^2-m^2-\mathbf{k}^2+i\epsilon]} \left[(k_0+\mu)\gamma^0 + m - \mathbf{k}\cdot\boldsymbol{\gamma}\right] \tag{A10}$$

and

$$\bar{S}^{0)}_{\mathrm{mat}}(k) = -2\pi\,\theta(|\mu|-|k_0|)\theta(-k_0\mu) \left[(k_0+\mu)\gamma^0 + m - \mathbf{k}\cdot\boldsymbol{\gamma}\right] \delta\left[(k_0+\mu)^2 - m^2 - \mathbf{k}^2\right] \tag{A11}$$

are the vacuum and matter parts, respectively. After integration of the proper time and making use of the identity in Eq. (A5), we find that this is identical to the usual free fermion propagator (8) in the absence of the field.

Expanding the expression in Eq. (A8) to linear order in magnetic field, we also easily obtain the following linear in B correction to the fermion propagator:

$$\begin{aligned}
\bar{S}^{(1)}(k) = \gamma^1\gamma^2 eB &\left\{ \int_0^{\infty} s ds\, e^{is[(k_0+\mu)^2-m^2-\mathbf{k}^2+i\epsilon]} + 2i\pi\theta(|\mu|-|k_0|)\theta(-k_0\mu)\delta'\left[(k_0+\mu)^2 - m^2 - \mathbf{k}^2\right] \right\} \\
&\times \left[(k_0+\mu)\gamma^0 + m - k^3\gamma^3\right].
\end{aligned} \tag{A12}$$

After integration over the proper time and making use of an identity, obtained from Eq. (A5) by differentiating with respect to $\mathcal{M}_n^2$, we obtain Eq. (9).

Appendix B: Calculation of the f_1, f_2, and f_3 terms

In this Appendix, we give the details of the calculation of the radiative corrections to axial current due to the f_1, f_2, and f_3 terms. We start from the general form of the result in Eq. (47) and calculate separately the two contributions, f_1 and $f_2 + f_3$. At the end we combine all contributions and calculate the final result for the $f_1 + f_2 + f_3$ contribution.

1. Calculation of f_1

Starting from the definition in Eq. (36), we find it convenient to rewrite the expression for f_1 in the following equivalent form:

$$f_1 \equiv f_1(m_\gamma) - f_1(\Lambda), \tag{B1}$$

where we took into account that the photon propagator is defined by Eq. (5), with Λ playing the role of the ultraviolet regulator. As follows from the definition,

$$
\begin{aligned}
f_1(m_\gamma) &= -64i\pi^2\alpha eB\frac{\partial}{\partial (m_c)^2}\int\frac{d^4pd^4k}{(2\pi)^8}\frac{\mu(p_0+\mu)-\mathbf{p}\cdot\mathbf{k}-2m^2}{[(P-K)^2-m_\gamma^2](P^2-m^2)}\delta\left[\mu^2-m_c^2-\mathbf{k}^2\right]\delta(k_0)\\
&= \frac{16i\pi\alpha eB}{k_F}\frac{\partial}{\partial k_F}\left[k_F\int\frac{p^2dpdp_0d\xi}{(2\pi)^5}\frac{\mu(p_0+\mu)-pk_F\xi-2m^2}{(p_0^2-p^2-k_F^2+2pk_F\xi-m_\gamma^2)[(p_0+\mu)^2-p^2-m^2]}\right],
\end{aligned}
\tag{B2}
$$

where we integrated over the energy k_0, the absolute value of the spacial momentum k, and all angular coordinates except for the angle θ_{kp} between $\mathbf{k}$ and $\mathbf{p}$. We also introduced the following short-hand notations: $k_F=\sqrt{\mu^2-m_c^2}$ and $\xi=\cos\theta_{kp}$. Note that the auxiliary quantities m_c and k_F should be replaced by the physical fermion mass m and the Fermi momentum $p_F=\sqrt{\mu^2-m^2}$, respectively, at the end of the calculation.

The integral over the energy p_0 can be calculated, using the following general result for the energy integration:

$$
i\int\frac{[X(p_0+\mu)\mu+Y]\,dp_0}{(p_0^2-b^2)\,[(p_0+\mu)^2-a^2]}=\frac{\pi}{b}\left[\frac{\theta(\mu-a)\,[X(b+\mu)\mu+Y]}{[(b+\mu)^2-a^2]}-\frac{\theta(a-\mu)\left[Xa\mu^2+(a+b)Y\right]}{a\,[(a+b)^2-\mu^2]}\right],
\tag{B3}
$$

where $a=\sqrt{p^2+m^2}$ and $b=\sqrt{p^2+k_F^2-2pk_F\xi+m_\gamma^2}$. Then we obtain

$$
\begin{aligned}
f_1(m_\gamma) = \frac{\alpha eB}{2\pi^3k_F}\frac{\partial}{\partial k_F}\int\frac{k_Fp^2dpd\xi}{b}&\left(\frac{\theta(p_F-p)\left[\mu(b+\mu)-pk_F\xi-2m^2\right]}{(b+\mu)^2-a^2}\right.\\
&\left.-\frac{\theta(p-p_F)\left[a\mu^2-(a+b)(pk_F\xi+2m^2)\right]}{a\,[(a+b)^2-\mu^2]}\right),
\end{aligned}
\tag{B4}
$$

The integral over the angular coordinate ξ can be easily performed, leading to the following result:

$$
\begin{aligned}
f_1(0) = \frac{\alpha eB}{4\pi^3k_F}\frac{\partial}{\partial k_F}\int pdp&\left[\theta(p_F-p)\left(p+k_F-|p-k_F|+\frac{\mu^2-3m^2-k_F^2}{2a}\ln\frac{(\mu+|p-k_F|+a)(\mu+p+k_F-a)}{(\mu+p+k_F+a)(\mu+|p-k_F|-a)}\right)\right.\\
&\left.+\theta(p-p_F)\left(p+k_F-|p-k_F|-\frac{2k_Fp}{a}+\frac{\mu^2-3m^2-k_F^2}{2a}\ln\frac{(a+|p-k_F|)^2-\mu^2}{(a+p+k_F)^2-\mu^2}\right)\right].
\end{aligned}
\tag{B5}
$$

Here, without loss of generality, we presented the result only for the case of the vanishing photon mass. This is justified because, as we will see below, the limit $m_\gamma\to 0$ does not produce any infrared singularities in the final result for f_1. If needed, an analogous expression for the case of a nonzero photon mass m_γ can be readily written down as well. It can be obtained from the above result for making the following three replacements: (i) $|p-k_F|\to\sqrt{(p-k_F)^2+m_\gamma^2}$, (ii) $p+k_F\to\sqrt{(p+k_F)^2+m_\gamma^2}$, and (iii) $\mu^2-3m^2-k_F^2\to\mu^2-3m^2-k_F^2-m_\gamma^2$ at two places in front of the logarithms.

After calculating the derivative with respect to k_F in Eq. (B5) and then substituting $k_F\to p_F$, we obtain

$$
\begin{aligned}
f_1(0) = \frac{\alpha eB}{4\pi^3p_F}\int pdp&\left[\theta(p_F-p)\left(\frac{2m^2p}{(p_F+\mu)(p_F^2-p^2)}-\frac{p_F}{a}\ln\frac{(\mu+p_F)^2-(p-a)^2}{(\mu+p_F)^2-(p+a)^2}\right)\right.\\
&\left.+\theta(p-p_F)\left(2-\frac{2p}{a}-\frac{p_F}{a}\ln\frac{p-p_F}{p+p_F}-\frac{2m^2p_F^2}{a(a+p)(p^2-p_F^2)}+\frac{2m^2p}{a(p^2-p_F^2)}\right)\right].
\end{aligned}
\tag{B6}
$$

It is easy to check that the above expression has a logarithmic ultraviolet divergency, i.e.,

$$
f_1^{\rm UV}(0)\simeq\alpha eB\frac{2\mu^2+m^2}{4\pi^3\sqrt{\mu^2-m^2}}\int\frac{dp}{p}.
\tag{B7}
$$

This confirms that an ultraviolet regularization is required in the calculation. As mentioned earlier, we utilize the Feynman regularization (B1), which is equivalent to using the photon propagator in Eq. (5). This is the same regularization, which is commonly used in the calculation of vacuum diagrams in QED, when the regularized expression is obtained from the divergent one by subtracting the contribution with a large photon mass Λ. In the case at hand,

therefore, we need the explicit expression for the function $f_1(\Lambda)$. The corresponding calculation is tedious, but straightforward. The result reads

$$
\begin{aligned}
f_1(\Lambda) = \frac{\alpha e B}{4\pi^3 p_F}\int p dp\Bigg[&\theta(p_F-p)\Bigg(\frac{p+p_F}{\sqrt{(p+p_F)^2+\Lambda^2}}+\frac{p-p_F}{\sqrt{(p-p_F)^2+\Lambda^2}}\\
&+\frac{(p_F-p)(2m^2+\Lambda^2)}{\sqrt{(p-p_F)^2+\Lambda^2}\left(2p_F^2-2p_Fp+\Lambda^2+2\mu\sqrt{(p-p_F)^2+\Lambda^2}\right)}\\
&-\frac{(p_F+p)(2m^2+\Lambda^2)}{\sqrt{(p+p_F)^2+\Lambda^2}\left(2p_F^2+2p_Fp+\Lambda^2+2\mu\sqrt{(p+p_F)^2+\Lambda^2}\right)}\\
&+\frac{p_F}{a}\ln\frac{\left(\mu+\sqrt{(p+p_F)^2+\Lambda^2}+a\right)\left(\mu+\sqrt{(p-p_F)^2+\Lambda^2}-a\right)}{\left(\mu+\sqrt{(p-p_F)^2+\Lambda^2}+a\right)\left(\mu+\sqrt{(p+p_F)^2+\Lambda^2}-a\right)}\Bigg)\\
&+\theta(p-p_F)\Bigg(\frac{p+p_F}{\sqrt{(p+p_F)^2+\Lambda^2}}+\frac{p-p_F}{\sqrt{(p-p_F)^2+\Lambda^2}}-\frac{2p}{a}\\
&+\frac{(2m^2+\Lambda^2)(p-p_F)\left(a+\sqrt{(p-p_F)^2+\Lambda^2}\right)}{a\sqrt{(p-p_F)^2+\Lambda^2}\left(2p^2-2pp_F+\Lambda^2+2a\sqrt{(p-p_F)^2+\Lambda^2}\right)}\\
&+\frac{(2m^2+\Lambda^2)(p+p_F)\left(a+\sqrt{(p+p_F)^2+\Lambda^2}\right)}{a\sqrt{(p+p_F)^2+\Lambda^2}\left(2p^2+2pp_F+\Lambda^2+2a\sqrt{(p+p_F)^2+\Lambda^2}\right)}\\
&+\frac{p_F}{a}\ln\frac{2p^2+2pp_F+\Lambda^2+2a\sqrt{(p+p_F)^2+\Lambda^2}}{2p^2-2pp_F+\Lambda^2+2a\sqrt{(p-p_F)^2+\Lambda^2}}\Bigg)\Bigg].
\end{aligned}
\tag{B8}
$$

Finally, as follows from the definition in Eq. (B1), the regularized expression of f_1 reads

$$
\begin{aligned}
f_1 = &\frac{\alpha e B}{4\pi^3 k_F}\int_0^{p_F} p dp\left(\frac{m^2}{(p_F-p)(p_F+\mu)}-\frac{m^2}{(p+p_F)(p_F+\mu)}+\frac{p_F}{a}\ln\frac{(\mu+p_F+p+a)(\mu+p_F-p-a)}{(\mu+p_F-p+a)(\mu+p_F+p-a)}\right)\\
&+\frac{\alpha e B}{4\pi^3 k_F}\int_{p_F}^{\infty} p dp\Bigg(2-\frac{p+p_F}{\sqrt{(p+p_F)^2+\Lambda^2}}-\frac{p-p_F}{\sqrt{(p-p_F)^2+\Lambda^2}}\\
&+\frac{m^2(a+p-p_F)}{a(p-p_F)(a+p)}-\frac{(2m^2+\Lambda^2)(p-p_F)\left(a+\sqrt{(p-p_F)^2+\Lambda^2}\right)}{a\sqrt{(p-p_F)^2+\Lambda^2}\left(2p^2-2pp_F+\Lambda^2+2a\sqrt{(p-p_F)^2+\Lambda^2}\right)}\\
&+\frac{m^2(a+p+p_F)}{a(p+p_F)(a+p)}-\frac{(2m^2+\Lambda^2)(p+p_F)\left(a+\sqrt{(p+p_F)^2+\Lambda^2}\right)}{a\sqrt{(p+p_F)^2+\Lambda^2}\left(2p^2+2pp_F+\Lambda^2+2a\sqrt{(p+p_F)^2+\Lambda^2}\right)}\\
&+\frac{p_F}{a}\ln\frac{p+p_F}{p-p_F}-\frac{p_F}{a}\ln\frac{2p^2+2pp_F+\Lambda^2+2a\sqrt{(p+p_F)^2+\Lambda^2}}{2p^2-2pp_F+\Lambda^2+2a\sqrt{(p-p_F)^2+\Lambda^2}}\Bigg)\Bigg|_{\Lambda\gg\mu}.
\end{aligned}
\tag{B9}
$$

Note that, in the first integral below the Fermi surface ($p \leq p_F$), we took the limit $\Lambda \to \infty$ because it does not cause any problem. It is essential, however, to keep Λ finite in the second integral above the Fermi surface ($p \geq p_F$).

A careful analysis of the regularized expression for f_1 in Eq. (B9) reveals a potentially serious problem: both integrals below and above the Fermi surface have infrared logarithmic divergencies, coming from from the regions near p_F. These divergencies cannot be avoided even when the photon mass is introduced as a regulator. (The divergencies do happen to vanish in the theory with massless fermions, $m = 0$, but this is of no importance as we discuss below.) Fortunately, as we will see below, the corresponding divergencies exactly cancel similar infrared divergencies in the expression for $f_2 + f_3$. Therefore, we come to the conclusion that the appearance of infrared divergencies in f_1, as well as in $f_2 + f_3$, is purely accidental and has no implications on physical observables. They can be viewed as a consequence of an ambiguous split of the finite expression $f_1 + f_2 + f_3$ into two separate contributions.

In order to carefully sort out the cancelation of the above mentioned (unphysical) infrared divergencies, it is useful to explicitly separate the divergent terms from regular ones in the corresponding expression for $f_1 = f_1^{(\mathrm{IR,div})} + f_1^{(\mathrm{IR,reg})}$. The divergent part of the expression reads

$$\begin{aligned} f_1^{(\mathrm{IR,div})} &= \frac{\alpha e B m^2}{4\pi^3 k_F}\left[\int_0^{k_F-\epsilon_1}\frac{pdp}{(k_F-p)(k_F+\mu)} + \int_{k_F+\epsilon_2}^{\infty} pdp\left(\frac{1}{(p-k_F)(a+p)} - \frac{1}{2p^2}\right)\right] \\ &= \frac{\alpha e B m^2}{4\pi^3}\left[\frac{1}{k_F+\mu}\left(\ln\frac{k_F}{\epsilon_1} - 1\right) + \frac{1}{k_F+\mu}\left(\ln\frac{\mu}{\epsilon_2} - \frac{3}{2}\right) + \frac{1}{2k_F}\left(\ln\frac{2k_F}{k_F+\mu} + \frac{1}{2}\right) + \frac{\mu+k_F}{m^2}\ln\frac{2\mu}{k_F+\mu}\right], \end{aligned} \tag{B10}$$

where we introduced infrared regulators ϵ_1 and ϵ_2 (with $\epsilon_1, \epsilon_2 \to 0$) that allow to deal with the problem in a rigorous way. Notice that, in the second integral we added a simple regular term, whose only purpose is to insure the ultraviolet convergence of the whole expression. The remaining regular part of the expression for f_1 reads

$$\begin{aligned} f_1^{(\mathrm{IR,reg})} &= \frac{\alpha e B}{4\pi^3 k_F}\int_0^{k_F} pdp\Bigg(-\frac{m^2}{(p+k_F)(k_F+\mu)} + \frac{k_F}{a}\ln\frac{(\mu+k_F+p+a)(\mu+k_F-p-a)}{(\mu+k_F-p+a)(\mu+k_F+p-a)}\Bigg) \\ &+ \frac{\alpha e B}{4\pi^3 k_F}\int_{k_F}^{\infty} pdp\Bigg(2 - \frac{p+k_F}{\sqrt{(p+k_F)^2+\Lambda^2}} - \frac{p-k_F}{\sqrt{(p-k_F)^2+\Lambda^2}} + \frac{m^2}{2p^2} + \frac{2m^2}{a(a+p)} + \frac{m^2}{(p+k_F)(a+p)} \\ &- \frac{(2m^2+\Lambda^2)(p-k_F)\left(a+\sqrt{(p-k_F)^2+\Lambda^2}\right)}{a\sqrt{(p-k_F)^2+\Lambda^2}\left(2p^2-2pk_F+\Lambda^2+2a\sqrt{(p-k_F)^2+\Lambda^2}\right)} \\ &- \frac{(2m^2+\Lambda^2)(p+k_F)\left(a+\sqrt{(p+k_F)^2+\Lambda^2}\right)}{a\sqrt{(p+k_F)^2+\Lambda^2}\left(2p^2+2pk_F+\Lambda^2+2a\sqrt{(p+k_F)^2+\Lambda^2}\right)} \\ &+ \frac{k_F}{a}\ln\frac{p+k_F}{p-k_F} - \frac{k_F}{a}\ln\frac{2p^2+2pk_F+\Lambda^2+2a\sqrt{(p+k_F)^2+\Lambda^2}}{2p^2-2pk_F+\Lambda^2+2a\sqrt{(p-k_F)^2+\Lambda^2}}\Bigg)\Bigg|_{\Lambda\gg\mu}. \end{aligned} \tag{B11}$$

Calculating the integrals in the case $m \ll |\mu|$, we arrive at the following results

$$f_1^{(\mathrm{IR,div})} \simeq \frac{\alpha e B m^2}{8\pi^3\mu}\left(\ln\frac{\mu}{\epsilon_1} + \ln\frac{\mu}{\epsilon_2} - 1\right), \tag{B12}$$

$$f_1^{(\mathrm{IR,reg})} \simeq \frac{\alpha e B\mu}{2\pi^3}\left(\ln\frac{\Lambda}{2\mu} + \frac{5}{4}\right) + \frac{\alpha e B m^2}{2\pi^3\mu}\ln\frac{\Lambda}{2\mu}. \tag{B13}$$

2. Calculation of $f_2 + f_3$

In this subsection, we calculate the expression for the sum $f_2 + f_3$ starting from the definition in Eqs. (40) and (44). As we we see below, the corresponding expression has no ultraviolet divergencies. Therefore, we could take the limit $\Lambda \to \infty$ in the expression for $f_2 + f_3$, i.e.,

$$\begin{aligned} f_2 + f_3 &= \frac{64 i\pi^2 \alpha e B}{3}\frac{\partial}{\partial(m_a)^2}\int\frac{d^4p d^4k}{(2\pi)^8}\frac{3(p_0+\mu)^2 - 3\mu(p_0+\mu) + p^2 - pk_F\xi + 3m^2}{\left[(P-K)^2 - m_\gamma^2\right](P^2 - m_a^2)}\delta\left(\mu^2 - m^2 - \mathbf{k}^2\right)\delta(k_0) \\ &= \frac{32 i\pi\alpha e B k_F}{3}\frac{\partial}{\partial(m_a)^2}\int\frac{p^2 dp dp_0 d\xi}{(2\pi)^5}\frac{4p^2 - pk_F\xi + 3m_a^2 + 3m^2 - 3\mu(p_0+\mu)}{(p_0^2 - p^2 - k_F^2 + 2pk_F\xi - m_\gamma^2)[(p_0+\mu)^2 - p^2 - m_a^2]}, \end{aligned} \tag{B14}$$

where we integrated over the energy k_0, the absolute value of the spacial momentum k, and all angular coordinates except for the angle θ_{kp} between $\mathbf{k}$ and $\mathbf{p}$. We also introduced an auxiliary quantity m_a, which should be replaced by the physical fermion mass m at the end of the calculation. It should be also noted that the some terms independent of m_a were dropped in the integrand of the last expression. This is justified because they vanish anyway after the derivative with respect to m_a^2 is calculated.

In order to calculate the integral over the energy p_0, we use again the result in Eq. (B3). Then, we arrive at

$$f_2 + f_3 = -\frac{\alpha e B}{6\pi^3}\frac{\partial}{\partial p_F}\int \frac{p^2 dp d\xi}{b}\left(\frac{\theta(p_F - p)\left[-3b\mu + 4p^2 - pk_F\xi - 3p_F^2 + 3m^2\right]}{(b+\mu)^2 - a^2}\right.$$
$$\left. - \frac{\theta(p - p_F)\left[3b\mu^2 + (a+b)(4p^2 - pk_F\xi - 3p_F^2 + 3m^2)\right]}{a\left[(a+b)^2 - \mu^2\right]}\right), \tag{B15}$$

where $b = \sqrt{p^2 + k_F^2 - 2pk_F\xi + m_\gamma^2}$ and $a = \sqrt{p^2 + \mu^2 - p_F^2}$. Note that, in this subsection, we distinguish the quantity $p_F \equiv \sqrt{\mu^2 - m_a^2}$ from the physical Fermi momentum $k_F = \sqrt{\mu^2 - m^2}$. After the partial derivative with respect to p_F is performed, the value of p_F will be replaced by k_F. Similarly, the auxiliary quantity a, which is a function of p_F, will be replaced by its physical counterpart $\sqrt{p^2 + m^2}$.

The integral over the angular coordinate ξ can be easily performed, leading to the following result:

$$f_2 + f_3 = -\frac{\alpha e B}{12\pi^3 k_F}\frac{\partial}{\partial p_F}\int p dp\left[\theta(p_F - p)\left(p + k_F - |p - k_F| + 4\mu\ln\frac{(\mu + |p - k_F|)^2 - a^2}{(\mu + p + k_F)^2 - a^2}\right.\right.$$
$$\left. + \frac{7(k_F^2 - p_F^2) + 8p^2 + 14m^2}{2a}\ln\frac{(\mu + |p - k_F| + a)(\mu + p + k_F - a)}{(\mu + p + k_F + a)(\mu + |p - k_F| - a)}\right)$$
$$+\theta(p - p_F)\left(p + k_F - |p - k_F| - \frac{2k_F p}{a} + 4\mu\ln\frac{(a + |p - k_F| + \mu)(a + p + k_F - \mu)}{(a + p + k_F + \mu)(a + |p - k_F| - \mu)}\right.$$
$$\left.\left. + \frac{7(k_F^2 - p_F^2) + 8p^2 + 14m^2}{2a}\ln\frac{(a + |p - k_F|)^2 - \mu^2}{(a + p + k_F)^2 - \mu^2}\right)\right], \tag{B16}$$

where, without loss of generality, we again took the limit of vanishing photon mass.

It is straightforward to show that the integral in Eq. (B16) has infrared logarithmic divergencies, similar to those in function f_1. Therefore, we follow the same kind of analysis as in the case of f_1 and extract explicitly the following infrared divergent terms:

$$f_2^{(\mathrm{IR,div})} + f_3^{(\mathrm{IR,div})} = -\frac{\alpha e B}{12\pi^3 k_F}\frac{\partial}{\partial p_F}\int p dp\left[\theta(p_F - \epsilon_1 - p)\left(-\frac{3m^2}{\mu}\ln\frac{(\mu + |p - k_F|)^2 - a^2}{(\mu + p + k_F)^2 - a^2}\right)\right.$$
$$\left. + \theta(p - p_F - \epsilon_2)\left(\frac{3m^2}{a}\ln\frac{a - \mu + p - k_F}{a + \mu + p + k_F} + \frac{3m^2(k_F + \mu)}{p^2}\right)\right]$$
$$= \frac{\alpha e B m^2}{4\pi^3}\left[\frac{1}{k_F + \mu}\left(\ln\frac{\mu\epsilon_1}{k_F(k_F + \mu)} - \frac{2k_F}{\mu}\right) + \frac{1}{k_F + \mu}\ln\frac{\epsilon_2}{k_F + \mu} + \frac{k_F + \mu}{k_F^2}\right]. \tag{B17}$$

Then, the leftover regular part reads

$$f_2^{(\mathrm{IR,reg})} + f_3^{(\mathrm{IR,reg})} = -\frac{\alpha e B}{12\pi^3 k_F}\frac{\partial}{\partial p_F}\int p dp\left[\theta(p_F - p)\left(p + k_F - |p - k_F| + 4\mu\ln\frac{(\mu + |p - k_F|)^2 - a^2}{(\mu + p + k_F)^2 - a^2}\right.\right.$$
$$\left. + \frac{7(k_F^2 - p_F^2) + 8p^2 + 14m^2}{2a}\ln\frac{(\mu + |p - k_F| + a)(\mu + p + k_F - a)}{(\mu + p + k_F + a)(\mu + |p - k_F| - a)} + \frac{3m^2}{\mu}\ln\frac{(\mu + |p - k_F|)^2 - a^2}{(\mu + p + k_F)^2 - a^2}\right)$$
$$+\theta(p - p_F)\left(p + k_F - |p - k_F| - \frac{2k_F p}{a} + 4\mu\ln\frac{(a + |p - k_F| + \mu)(a + p + k_F - \mu)}{(a + p + k_F + \mu)(a + |p - k_F| - \mu)}\right.$$
$$\left.\left. + \frac{7(k_F^2 - p_F^2) + 8p^2 + 14m^2}{2a}\ln\frac{(a + |p - k_F|)^2 - \mu^2}{(a + p + k_F)^2 - \mu^2} - \frac{3m^2}{a}\ln\frac{a - \mu + p - k_F}{a + \mu + p + k_F} - \frac{3m^2(k_F + \mu)}{p^2}\right)\right] \tag{B18}$$

After calculating the derivative with respect to p_F in the regular piece (note that $a = \sqrt{p^2 + \mu^2 - p_F^2}$, i.e., a is a

function of p_F) and substituting $p_F \to k_F$ afterwards, we obtain

$$
\begin{aligned}
f_2^{(\mathrm{IR,reg})} + f_3^{(\mathrm{IR,reg})} &= -\frac{\alpha e B}{2\pi^3}\left(\frac{k_F^2}{3\mu} + \frac{m^2}{\mu}\ln\frac{k_F^2}{\mu(\mu+k_F)} + \frac{m^2(k_F+\mu)}{2k_F^2}\right) \\
&+ \frac{\alpha e B}{4\pi^3}\int\frac{p dp}{a^2}\left[\theta(k_F-p)\left(\frac{2m^2 p}{\mu(k_F+\mu)} + \frac{p^2}{a}\ln\frac{k_F(k_F+\mu)+p(a-p)}{k_F(k_F+\mu)-p(a+p)}\right)\right. \\
&\left.+\theta(p-k_F)\left(\frac{2pk_F}{3a} + \frac{8ak_F}{3(a+p)} - \frac{m^2(a-p)}{ak_F+p\mu} + \frac{p^2}{a}\ln\frac{p-k_F}{p+k_F} + \frac{m^2}{a}\ln\frac{a+p-k_F-\mu}{a+p+k_F+\mu}\right)\right]
\end{aligned}
\tag{B19}
$$

As is easy to check, the integrand is $\propto 1/p^3$ when $p \to \infty$. It is clear, therefore, that the expression is convergent and no additional ultraviolet regularization is needed. Integrating over the momentum, we finally obtain

$$
\begin{aligned}
f_2^{(\mathrm{IR,reg})} + f_3^{(\mathrm{IR,reg})} &= -\frac{\alpha e B k_F}{2\pi^3}\left[\frac{k_F}{3\mu} + \frac{m^2}{k_F\mu}\ln\frac{k_F^2}{\mu(\mu+k_F)} + \frac{m^2(k_F+\mu)}{2k_F^3}\right] \\
&+ \frac{\alpha e B}{2\pi^3}\left[k_F\left(1 + \frac{m^2}{\mu(k_F+\mu)} - \frac{\mu^2+m^2}{k_F\mu}\ln\frac{k_F+\mu}{\mu}\right)\right. \\
&\left.+\mu\left(\frac{\mu-k_F}{k_F+\mu}\left(1+\frac{k_F m^2}{3\mu^2(k_F+\mu)}\right) + \frac{k_F(k_F+\mu)}{2\mu^2}\ln\frac{\mu}{k_F} - \left(2+\frac{k_F}{\mu}\right)\ln\frac{2\mu}{k_F+\mu} + \ln 2 - 1\right)\right]
\end{aligned}
\tag{B20}
$$

For $m \ll |\mu|$, the final expressions for the infrared divergent and regular contributions simplify as follows:

$$
f_2^{(\mathrm{IR,div})} + f_2^{(\mathrm{IR,div})} \simeq \frac{\alpha e B m^2}{2\pi^3\mu}\left[\frac{1}{4}\left(\ln\frac{\epsilon_1}{2\mu} - 2\right) + \frac{1}{4}\ln\frac{\epsilon_2}{2\mu} + 1\right], \tag{B21}
$$

$$
f_2^{(\mathrm{IR,reg})} + f_3^{(\mathrm{IR,reg})} \simeq -\frac{\alpha e B\mu}{6\pi^3} - \frac{\alpha e B m^2}{24\pi^3\mu}. \tag{B22}
$$

3. Collecting all contributions

As seen from Eq. (47), the final expression for the axial current is given in terms of the sum $f_1 + f_2 + f_3$. The corresponding function is obtained by collecting all the divergent and regular terms, calculated in the previous two subsections of this Appendix. In the case $m \ll |\mu|$, in particular, the result reads

$$
\begin{aligned}
f_1 + f_2 + f_3 &= f_1^{(\mathrm{IR,div})} + f_2^{(\mathrm{IR,div})} + f_2^{(\mathrm{IR,div})} + f_1^{(\mathrm{IR,reg})} + f_2^{(\mathrm{IR,reg})} + f_3^{(\mathrm{IR,reg})} \\
&\simeq \frac{\alpha e B\mu}{2\pi^3}\left(\ln\frac{\Lambda}{2\mu} + \frac{11}{12}\right) + \frac{\alpha e B m^2}{2\pi^3\mu}\left(\ln\frac{\Lambda}{2^{3/2}\mu} + \frac{1}{6}\right).
\end{aligned}
\tag{B23}
$$

Notice that all infrared regulators (ϵ_1 and ϵ_2), which were introduced in the divergent parts of f_1 and $f_2 + f_3$ cancelled out. The only regulator in the last expression is the ultraviolet one Λ. In the final expression for the axial current (55), this dependence on the ultraviolet regulator cancels out exactly with a similar dependence coming from the counterterms contribution in Eq. (52).

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[1] D. Kharzeev, Phys. Lett. B **633**, 260 (2006).
[2] D. Kharzeev and A. Zhitnitsky, Nucl. Phys. A **797**, 67 (2007).
[3] D. E. Kharzeev, L. D. McLerran and H. J. Warringa, Nucl. Phys. A **803**, 227 (2008).
[4] K. Fukushima, D. E. Kharzeev and H. J. Warringa, Phys. Rev. D **78**, 074033 (2008).
[5] K. Fukushima, arXiv:1209.5064 [hep-ph], published in "Strongly Interacting Matter in Magnetic Fields", edited by D. Kharzeev, K. Landsteiner, A. Schmitt, H.-U. Yee. Lect. Notes Phys. **871** (Springer, Berlin, 2013) pp. 241-259.
[6] B. I. Abelev *et al.* [STAR Collaboration], Phys. Rev. Lett. **103**, 251601 (2009); Phys. Rev. C **81**, 054908 (2010).
[7] L. Adamczyk *et al.* [STAR Collaboration], arXiv:1303.0901 [nucl-ex].
[8] G. Wang [STAR Collaboration], arXiv:1210.5498 [nucl-ex].
[9] H. Ke [STAR Collaboration], J. Phys. Conf. Ser. **389**, 012035 (2012).

[10] I. Selyuzhenkov [ALICE Collaboration], Prog. Theor. Phys. Suppl. **193**, 153 (2012).
[11] S. A. Voloshin, Phys. Rev. C **70**, 057901 (2004).
[12] D. E. Kharzeev, Annals Phys. **325**, 205 (2010); K. Fukushima, D. E. Kharzeev and H. J. Warringa, Nucl. Phys. A **836**, 311 (2010).
[13] A. Vilenkin, Phys. Rev. D **22**, 3067 (1980).
[14] M. A. Metlitski and A. R. Zhitnitsky, Phys. Rev. D **72**, 045011 (2005).
[15] G. M. Newman and D. T. Son, Phys. Rev. D **73**, 045006 (2006).
[16] E. V. Gorbar, V. A. Miransky and I. A. Shovkovy, Phys. Rev. D **83**, 085003 (2011).
[17] Y. Burnier, D. E. Kharzeev, J. Liao and H. -U. Yee, Phys. Rev. Lett. **107**, 052303 (2011); arXiv:1208.2537 [hep-ph].
[18] S. L. Adler, Phys. Rev. **177**, 2426 (1969); J. S. Bell and R. Jackiw, Nuovo Cim. A **60**, 47 (1969).
[19] S. L. Adler and W. A. Bardeen, Phys. Rev. **182**, 1517 (1969).
[20] G. Basar and G. V. Dunne, arXiv:1207.4199 [hep-th], published in "Strongly Interacting Matter in Magnetic Fields", edited by D. Kharzeev, K. Landsteiner, A. Schmitt, H.-U. Yee. Lect. Notes Phys. **871** (Springer, Berlin, 2013) pp. 261-294.
[21] E. V. Gorbar, V. A. Miransky and I. A. Shovkovy, Phys. Rev. C **80**, 032801(R) (2009).
[22] E. V. Gorbar, V. A. Miransky and I. A. Shovkovy, Phys. Lett. B **695**, 354 (2011).
[23] K. Fukushima and M. Ruggieri, Phys. Rev. D **82**, 054001 (2010).
[24] M. E. Peskin and D. V. Schroeder, "An Introduction To Quantum Field Theory," (Westview Press, 1995) 842 pages.
[25] J. S. Schwinger, Phys. Rev. **82**, 664 (1951).
[26] E. C. G. Stueckelberg, Helv. Phys. Acta **30**, 209 (1957).
[27] E. V. Gorbar, V. A. Miransky, I. A. Shovkovy, and Xinyang Wang, work in preparation.
[28] J.-H. Gao, Z.-T. Liang, S. Pu, Q. Wang and X.-N. Wang, Phys. Rev. Lett. **109**, 232301 (2012); J.-W. Chen, S. Pu, Q. Wang and X.-N. Wang, arXiv:1210.8312 [hep-th].
[29] S. Weinberg, "The Quantum Theory of Fields. Vol. 1: Foundations", (University Press, Cambridge, 1995) pp. 564-596.
[30] T. Kinoshita, J. Math. Phys. **3**, 650 (1962).
[31] T. D. Lee and M. Nauenberg, Phys. Rev. **133**, B1549 (1964).
[32] P. I. Fomin, V. A. Miransky, and Y. A. Sitenko, Phys. Lett. B **64**, 444 (1976).
[33] In massless QED without external electromagnetic fields, fermions interact through neutral vector currents [32], despite the vanishing renormalized electric charge. In fact, massless QED yields the simplest example of an unparticle gauge field theory [34], in which the infrared fixed point is Gaussian. There are no one-particle asymptotic states in unparticle theories. Instead, the asymptotic states are described by fermionic and bosonic jets [31, 32, 34].
[34] H. Georgi, Phys. Rev. Lett. **98**, 221601 (2007).